Irrational Seshadri constants from dihedral orbits

Antonio Laface

Universidad de Concepción
Joint work with Luca Ugaglia

Seshadri constants measure local positivity

Setting. S a normal complex projective surface, L an ample Cartier divisor, $p_1, \dots, p_r \in S$ distinct smooth points, $\pi\colon \mathrm{Bl}_{p_1,\dots,p_r} S \to S$ the blow-up with exceptional curves $E_1, \dots, E_r$.

Definition (multipoint Seshadri constant)

$$\varepsilon(L; p_1, \dots, p_r) \;=\; \inf_C \frac{L \cdot C}{\sum_i \mathrm{mult}_{p_i} C} \;=\; \sup\Big\{ t \geq 0 :\ \pi^* L - t \textstyle\sum_i E_i \text{ is nef} \Big\},$$

the infimum over integral curves C through at least one p_i.

Notation: $\varepsilon(L; r)$ is the value at **very general** r-tuples of points.

An irrational value can only sit at the volume bound

A nef class has nonnegative square, so if $\pi^*L - t\sum E_i$ is nef then $L^2 - rt^2 \geq 0$:

$$\varepsilon(L; p_1, \ldots, p_r) \;\leq\; \sqrt{L^2/r}.$$

Irrationality criterion

$$\varepsilon(L;p) \notin \mathbb{Q} \quad \Longleftrightarrow \quad L^2 \text{ is not a square and } \pi^*L - \sqrt{L^2}\,E \text{ is nef.}$$

Where the question comes from

- **1959** — **Nagata** [Nag59]. For $r \geq 10$ very general points of $\mathbb{P}^2$,
$$\varepsilon\Big(\mathcal{O}_{\mathbb{P}^2}(1); r\Big) = \frac{1}{\sqrt{r}},$$
i.e. every plane curve of degree d satisfies $d \geq \frac{1}{\sqrt{r}} \sum_i \operatorname{mult}_{p_i} C$. Nagata proved it when $r = s^2$ with $s \geq 4$.

- **1992** — **Demailly** [Dem92]. Introduced Seshadri constants as a measure of **local positivity**: $\varepsilon(L; p)$ is the asymptotic rate at which sections of kL separate jets at p, as $k \to \infty$.

Irrational examples were known only conditionally

- **2016** — **Dumnicki, Küronya, Maclean, Szemberg** [DKMS16]. Irrational one-point constants, *assuming* the Segre–Harbourne–Gimigliano–Hirschowitz (SHGH) conjecture.
- **2018** — **Hanumanthu, Harbourne** [HH18]. Irrational one-point constants, *assuming* that every negative curve on the blow-up at the evaluation point is a (-1)-curve.
- **2025** — **Dionne, Roth** [DR25]. Reduced $\varepsilon\Big(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1);10\Big) = 1/\sqrt{5}$ to the nefness of one explicit class R with $R^2 = 0$.

2026: the first unconditional results

- **Laface–Ugaglia** [LU26]. An ample divisor on a smooth surface with an **irrational one-point** Seshadri constant, built from orbits of a fixed dihedral group (see [MMSZ26] for a generalization to D_n-actions with odd $n \geq 5$). *This talk.*

- **OpenAI** [OAI26a]. Proof of Nagata's conjecture; exposition by Tevelev [Tev26].

- **OpenAI** [OAI26b, OAI26c]. For every smooth complex polarized variety (X, L) of dimension $n \geq 2$ there is $r_0 = r_0(X, L)$ such that

$$\varepsilon(L; r) = \left(\frac{L^n}{r}\right)^{1/n} \qquad \text{for all } r \geq r_0;$$

  the same holds in positive characteristic [OAI26d].

Main theorem

Theorem [LU26]

Let X_9 be the blow-up of $\mathbb{P}^2$ at nine very general points, with line class H and exceptional curves $E_1, \ldots, E_9$. The divisor

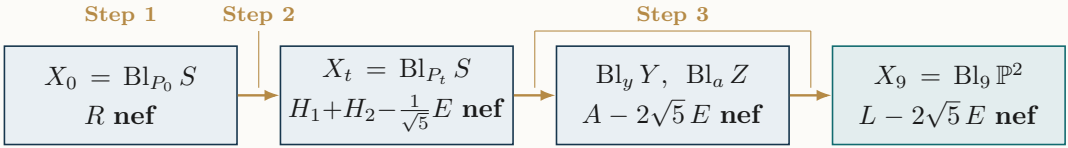
$$L = 9H - 3(E_1 + \cdots + E_5) - 2(E_6 + \cdots + E_9)$$

is ample, and for a very general point $p \in X_9$

$$\varepsilon(L; p) = \sqrt{L^2} = 2\sqrt{5}.$$

Optimality: on blow-ups of $\mathbb{P}^2$ at ≤ 8 points, one-point Seshadri constants of ample line bundles are rational [HH18].

Strategy: one irrational nef class, transported



1. **Step 1.** A birational self-map of $S := \mathbb{P}^1 \times \mathbb{P}^1$ acts on $\mathrm{Pic}\, X_0$ by a matrix T; normalised images of a ruling converge to an irrational nef class R.
2. **Step 2.** Moving the ten points $P_0 = G\cdot(1,1) = \{p_1, \dots, p_{10}\}$, with $G \simeq D_5$, off the curve Γ turns R into $H_1 + H_2 - \frac{1}{\sqrt{5}}E$, i.e. $\varepsilon\Big(\mathcal{O}(1,1); P_t\Big) = 1/\sqrt{5}$.
3. **Step 3.** Pass to $Y = S/G$, resolve to Z, view Z as a plane blow-up, and move the nine centres to very general position.

Ten points on a rational curve, permuted by D_5

Let $S = \mathbb{P}^1 \times \mathbb{P}^1$ with affine coordinates (x, z). Fix $u > 1$ and a primitive fifth root of unity ζ, and let $G \simeq D_5$ be generated by

$$r(x, z) = (\zeta x,\ \zeta^3 z), \qquad s(x, z) = (u/x,\ u^3/z).$$

- The rational curve $\Gamma = \{z = x^3\}$ is G-invariant.
- Hence the orbit $P_0 = G \cdot (1, 1)$ consists of **ten points on** Γ.

Ten points on a rational curve, permuted by D_5

Notation: on every blow-up $X = \mathrm{Bl}_P S$ at a ten-point orbit $H_1 = [x = \text{const}]$, $H_2 = [z = \text{const}]$, $E = E_1 + \cdots + E_{10}$, with

$$H_1^2 = H_2^2 = 0, \quad H_1 \cdot H_2 = 1, \quad E^2 = -10.$$

On $X_0 = \mathrm{Bl}_{P_0} S$: the strict transform Γ_0 of Γ has class

$$\gamma = 3H_1 + H_2 - E, \quad \gamma^2 = 6 - 10 = -4.$$

Each dihedral orbit produces the next one

Starting from a ten-point orbit $P_j \subset S$, set $X_j = \mathrm{Bl}_{P_j} S$.

- The ruling $|H_2|$ and the pencil $|H_1 + 5H_2 - E|$ of $(1,5)$-curves through P_j give a morphism $\varphi_j \colon X_j \to S$.
- φ_j contracts the ten curves of class $H_2 - E_i$ (the H_2-fibres through the points) to a new ten-point set $P_{j+1} \subset S$.
- Hence $\varphi_j = \pi_{j+1} \circ f_j$ for an isomorphism f_j onto $X_{j+1} = \mathrm{Bl}_{P_{j+1}} S$.

$$\begin{array}{ccc} X_j & \xrightarrow[\sim]{f_j} & X_{j+1} \\ {\scriptstyle \pi_j}\downarrow & \searrow^{\varphi_j} & \downarrow{\scriptstyle \pi_{j+1}} \\ S & \xrightarrow{\quad F_j \quad} & S \end{array}$$

P_{j+1} is again a dihedral orbit, so the construction **repeats for all** $j \geq 0$.

On Picard groups every step acts by the same matrix T

Reading off the construction, the pullback f_j^* acts as

$$\begin{aligned} H_1 &\longmapsto H_2 && \text{(the two rulings are swapped)} \\ H_2 &\longmapsto H_1 + 5H_2 - E && \text{(the pencil of } (1,5)\text{-curves)} \\ E_i &\longmapsto H_2 - E_i && \text{(the contracted fibres)} \end{aligned}$$

In the basis (H_1, H_2, E) this is the isometry

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 5 & 10 \\ 0 & -1 & -1 \end{pmatrix}, \qquad (f_{j-1} \circ \cdots \circ f_0)^* = T^j.$$

$T^j(H_2)$ is the pullback of a ruling of X_j under an isomorphism, so $T^j(H_2)$ **is nef on** X_0 **for every** $j \geq 0$.

The normalised limit is an irrational nef class R

$\det(T-\lambda) = -(\lambda-1)(\lambda^2-3\lambda+1)$, so the dominant eigenvalue is $\lambda = \frac{3+\sqrt{5}}{2}$, with eigenvector

$$R = \frac{5-\sqrt{5}}{2}\,H_1 + \frac{5+\sqrt{5}}{2}\,H_2 - E, \qquad R^2 = 0.$$

The other eigenvalues are 1 and $\lambda^{-1} < 1$, so $\lambda^{-j}T^j(H_2) \to c\,R$ with $c > 0$. The nef cone is closed.

R is nef on X_0, and $R \cdot \gamma = \sqrt{5}$.

Statement: a balanced (-4)-curve reflects nef cones

Let $\mathcal{X} \to B$ be a smooth family of surfaces over a smooth curve, with central fibre X_0, and **marked**: $\operatorname{Pic} X_t$ is identified with $\operatorname{Pic} X_0$ for all t (here: H_1, H_2, E_i on every fibre).

Reflection lemma [Pet14, DR25]

Let $\Gamma_0 \simeq \mathbb{P}^1 \subset X_0$ with $\Gamma_0^2 = -4$ and **balanced** normal bundle in the threefold, $N_{\Gamma_0/\mathcal{X}} \simeq \mathcal{O}(-2) \oplus \mathcal{O}(-2)$. Put $\gamma = [\Gamma_0]$ and define the isometry (reflection)

$$s_\gamma(v) = v + \frac{v \cdot \gamma}{2}\,\gamma.$$

Then $s_\gamma\Big(\operatorname{Nef} X_0\Big) \subseteq \operatorname{Nef} X_t$ for very general $t \in B$.

Proof (1): limits of curves contain Γ_0 with multiplicity μ

Let C be an integral curve on the generic fibre, $\mathcal{C} \subset \mathcal{X}$ its closure, and $C_0 = \mathcal{C}|_{X_0}$ its limit: an effective divisor with $[C_0] = [C] =: c$.

Near a point of Γ_0 choose coordinates (z, s, t) with $X_0 = \{t = 0\}$ and $\Gamma_0 = \{s = t = 0\}$. If $\mathcal{C}$ has multiplicity μ along Γ_0, its equation is

$$F(z,s,t) = \underbrace{\sum_{i+j=\mu} a_{ij}(z)\, s^i t^j}_{F_\mu \not\equiv 0} + \sum_{i+j>\mu} a_{ij}(z)\, s^i t^j.$$

$\mathcal{C}$ does not contain X_0, so $F(z, s, 0) \not\equiv 0$; and every term surviving $t = 0$ has $j = 0$, hence $i \geq \mu$. So $s^\mu \mid F(z, s, 0)$:

$$\operatorname{mult}_{\Gamma_0} C_0 \geq \mu, \qquad \text{i.e.} \qquad C_0 - \mu\,\Gamma_0 \text{ is effective.}$$

Proof (2): the normal bundle forces μ to be large

Let $\mathcal{I} = (s,t)$ be the ideal of Γ_0 in $\mathcal{X}$, and write $\mathcal{O}(k) = \mathcal{O}_{\mathbb{P}^1}(k)$. Then $\mathcal{I}/\mathcal{I}^2 = N^\vee \simeq \mathcal{O}(2)^{\oplus 2}$. The leading form F_μ is a **nonzero global section** of

$$\Big(\mathcal{I}^\mu/\mathcal{I}^{\mu+1}\Big) \otimes \mathcal{O}_{\Gamma_0}(\mathcal{C}) \;\simeq\; \operatorname{Sym}^\mu N^\vee \otimes \mathcal{O}_{\Gamma_0}(\mathcal{C}) \;\simeq\; \mathcal{O}\Big(c\cdot\gamma + 2\mu\Big)^{\oplus(\mu+1)}.$$

Balancedness is used here: every summand has the *same* degree. A line bundle on $\mathbb{P}^1$ with a nonzero section has degree ≥ 0, so

$$c\cdot\gamma + 2\mu \geq 0.$$

Petrakiev's estimate: $\operatorname{mult}_{\Gamma_0} C_0 \;\geq\; \mu \;\geq\; -\dfrac{c\cdot\gamma}{2}.$

Proof (3): reflected curve classes stay effective on X_0

For C as before, split the reflected class as

$$s_\gamma(c) = c + \frac{c\cdot\gamma}{2}\,\gamma = \underbrace{\left(C_0 - \mu\Gamma_0\right)}_{\text{effective by (1)}} + \underbrace{\left(\mu + \frac{c\cdot\gamma}{2}\right)}_{\geq 0 \text{ by (2)}}\Gamma_0,$$

an **effective $\mathbb{Q}$-divisor on** X_0. Hence, for $D \in \operatorname{Nef} X_0$,

$$s_\gamma(D)\cdot c \;=\; D\cdot s_\gamma(c) \;\geq\; 0 \qquad (s_\gamma \text{ is an isometry with } s_\gamma^2 = \mathrm{id}).$$

For very general t, every curve on X_t arises from the generic fibre, so $s_\gamma(D)$ is nef on X_t. □

Application: move the orbit, keep the group

Keep G fixed and move the base point along a curve $t \mapsto p_t$ with $p_0 = (1,1)$. Blowing up the ten curves traced by $P_t = G \cdot p_t$ in $S \times B$ gives $\mathcal{X} \to B$; let Γ_0 be the strict transform of $\Gamma \times \{0\}$.

Before blowing up, $N_{\Gamma\times\{0\}/S\times B} = \mathcal{O}(6) \oplus \mathcal{O}$, since $\Gamma^2 = 6$. Each point $p_i \in P_0$ moves with a velocity whose normal component is $v_i \in N_{\Gamma/S}|_{p_i}$. Blowing up performs an elementary transformation at each p_i [Pet14]:

$$0 \to N \to \mathcal{O}(6) \oplus \mathcal{O} \xrightarrow{(f,g) \mapsto (f(p_i) - v_i\, g(p_i))_i} \bigoplus_{i=1}^{10} \mathbb{C}_{p_i} \to 0.$$

N has rank 2 and degree $6 - 10 = -4$, so

$$N \simeq \mathcal{O}(-2)^{\oplus 2} \iff H^0\Big(N(1)\Big) = 0.$$

Balancedness is a 10×10 rank condition

Twist the sequence by $\mathcal{O}(1)$ and take global sections: $H^0\big(N(1)\big)$ is the kernel of the evaluation map

$$M\colon \underbrace{H^0\big(\mathcal{O}(7)\big)}_{\dim 8} \oplus \underbrace{H^0\big(\mathcal{O}(1)\big)}_{\dim 2} \longrightarrow \mathbb{C}^{10}, \qquad (f, g) \longmapsto \big(f(p_i) - v_i\, g(p_i)\big)_{i=1}^{10}.$$

M is a square 10×10 matrix, and

$$\operatorname{rk} M = 10 \iff H^0\big(N(1)\big) = 0 \iff N \simeq \mathcal{O}(-2)^{\oplus 2}.$$

For the trajectory $p_t = (1 + t, 1)$ a direct computation gives $\operatorname{rk} M = 10$: Γ_0 **is balanced**, and the reflection lemma applies.

Reflecting R gives exactly the class we want

Apply the lemma to $R \in \operatorname{Nef} X_0$, using $R \cdot \gamma = \sqrt{5}$:

$$s_\gamma(R) = R + \frac{\sqrt{5}}{2}\,\gamma = \left(\frac{5}{2} + \sqrt{5}\right)\left(H_1 + H_2 - \frac{1}{\sqrt{5}}E\right).$$

For a very general free G-orbit $P \subset \mathbb{P}^1 \times \mathbb{P}^1$,

$$\varepsilon\Big(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1); P\Big) = \frac{1}{\sqrt{5}}.$$

The ten-point statement becomes a one-point statement

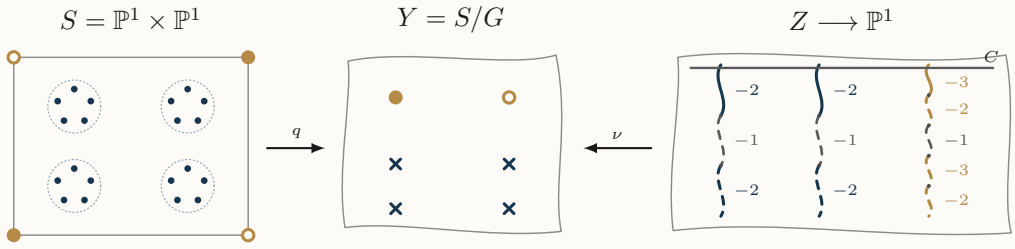
For a very general free orbit P, set $y = q(P)$ and $a = \nu^{-1}(y)$. Here q is the quotient, ν the minimal resolution, and the vertical arrows are blow-ups:

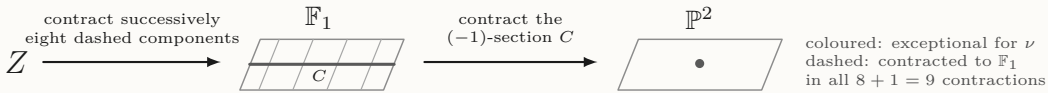
$$\begin{array}{ccccc} \mathrm{Bl}_P S & \xrightarrow{\widetilde{q}} & \mathrm{Bl}_y Y & \xleftarrow{\widetilde{\nu}} & \mathrm{Bl}_a Z \\ \downarrow{\scriptstyle \pi_P} & & \downarrow{\scriptstyle \pi_y} & & \downarrow{\scriptstyle \pi_a} \\ S & \xrightarrow{q} & Y & \xleftarrow{\nu} & Z \end{array}$$

The divisor $-5K_S$ descends to an ample Cartier divisor A with $A^2 = 20$, and $\widetilde{q}^*(\pi_y^* A - 2\sqrt{5}\, E_y)$ is ten times the nef class already constructed; hence

$$\widetilde{\nu}^*(\pi_y^* A - 2\sqrt{5}\, E_y) \quad \text{is nef on } \mathrm{Bl}_a Z.$$

Z is a blow-up of $\mathbb{P}^2$ at nine (infinitely near) points





Deform the nine centres to very general position

Deforming the nine blow-up centres of Z to very general position gives X_9. In a suitable plane model, ν^*A becomes

$$L = 2(-K_{X_9}) + \eta^*(-K_{X_5}), \qquad L^2 = 20,$$

where $\eta\colon X_9 \to X_5$ contracts $E_6, \dots, E_9$ to a del Pezzo surface of degree four. Since $-K_{X_9}$ is nef and $-K_{X_5}$ is ample, L is positive on every noncontracted curve. On the four contracted curves, $L \cdot E_i = 2$. Thus L **is ample** by Nakai–Moishezon.

The class $\pi_p^*L - 2\sqrt{5}\,E_p$ remains nef for very general $p \in X_9$. This gives

$$\varepsilon(L; p) = 2\sqrt{5}.$$

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